STABILITY OF SHELLS IN CREEP

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We consider a thin homogeneous shell subjected to an arbitrary load causing loss of stability. We assume that the shell has some initial irregularities in its middle surface which can be described in terms of certain initial displacements. When the load is applied, these initial irregularities begin to develop due to creep and cause a redistribution of stresses over both the thickness and the entire area of the shell. This process of stress redistribution may be so considerable that at a certain moment the equilibrium state of the shell may become unstable in Euler's sense, i.e., at a certain moment several modes of equilibrium may be possible, transition to any one of these being instantaneous. We shall call this moment the "critical moment" of loss of stability of the shell.

The deviation of the subcritical stress and strain state of an actual shell from the basic state corresponding to a perfectly smooth shell can be described by a system of equations in the stress and deflection functions, assuming that the quantities characterizing these deviations satisfy linearlized creep relations analogous to the relations for visco-elastic bodies. This system of equations must be combined with a system of stability equations which takes into account the stresses and strains defined by the system of equations of the subcritical state.

§1. **Formulation of the problem.** We assume that the subcritical stress components can be represented by

$$\sigma_{mn} = \sigma_{mn}^{\circ} + \frac{1}{h} T_{mn} + \frac{12}{h^3} M_{mn} z\,. \tag{1.1}$$

Here σ_{mn}° are the stress components of the basic state; T_{mn} and M_{mn} are the additional specific forces and moments referred to the middle surface of the shell; z is the distance from the middle surface; and h is the thickness of the shell.

It should be pointed out that, strictly speaking, in the case of creep, when the physical relations are nonlinear, the assumption of a linear stress distribution over the shell thickness contradicts the hypothesis of straight normals. But if the subcritical state of the shell differs only slightly from the basic state σ_{mn}° corresponding to a perfectly smooth shell, the physical relations can be linearized with respect to the basic state. This eliminates the contradiction. In the process of creep the equilibrium equations of the subcritical state take the form

$$\frac{\partial T_{m1}}{\partial x} + \frac{\partial T_{m2}}{\partial y} = 0, \qquad \frac{\partial M_{m1}}{\partial x} + \frac{\partial M_{m2}}{\partial y} = Q_m\,, \tag{1.2}$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{T_{11}}{R_{11}} + \frac{T_{22}}{R_{22}} - T_{11}\varkappa_{11} - T_{22}\varkappa_{22} - 2T_{12}\varkappa_{12} = 0 \quad (m = 1,\,2). \tag{1.3}$$

Here Q_m is the specific shear force, $\varkappa_{mn}$ are the parameters of variation of curvature and twist of the middle surface; and R_{mn} is the radius of the shell.

The system of equations (1.2) and (1.3) with the associated relations between stress and strain describes the process of buckling of the shell in time and the process of redistribution of stresses. Suppose that at a certain moment the shell becomes unstable, i.e., the shell instantaneously goes over into the neighboring equilibrium state characterized by forces T_{mn}, $+T_{mn}{}^*$, $Q_m + Q_m{}^*$, moments $M_{mn} + M_{mn}{}^*$, and curvatures $\varkappa_{mn} + \varkappa_{mn}{}^*$. Setting up the equilibrium equations for this neighboring state and subtracting Eqs. (1.2) and (1.3), we obtain the stability equations

$$\frac{\partial T_{m_1}^*}{\partial x} + \frac{\partial T_{m_2}^*}{\partial y} = 0, \quad \frac{\partial M_{m_1}^*}{\partial x} + \frac{\partial M_{m_2}^*}{\partial y} = Q_m{}^*\,. \tag{1.4}$$

$$\begin{gathered}\frac{\partial Q_1{}^*}{\partial x} + \frac{\partial Q_2{}^*}{\partial y} + \frac{T_{11}{}^*}{R_{11}} + \frac{T_{22}{}^*}{R_{22}} - T_{11}\varkappa_{11}{}^* - 2T_{12}\varkappa_{12}{}^* - \\ - T_{22}\varkappa_{22}{}^* - T_{11}{}^*\varkappa_{11} - 2T_{12}{}^*\varkappa_{12} - T_{22}{}^*\varkappa_{22} = 0 \\ (m = 1,2)\end{gathered} \tag{1.5}$$

where the nonlinear terms have been discarded.

§2. **Physical relations.** Equations (1.2)—(1.5) are not a closed system of equations for the problem posed, because they must be combined with equations determining the relationship between the stress and strain components during creep and at the moment of loss of stability together with the relations between the strain components and the displacement components.

Let the equation of state for creep be

$$\dot{p}_i = g(\sigma_i,\, p_i)\,\sigma_i\,, \tag{2.1}$$

while the components of the creep strain rate tensor $\dot{p}_{mn}$ and the components of the stress deviator s_{mn} satisfy the relations of flow theory

$$\begin{gathered}\dot{p}_{mn} = \frac{3}{2}\, g(\sigma_i,\, p_i)\, s_{mn}\,, \\ p_{mn} = \varepsilon_{mn} - \frac{3}{2E}\, s_{mn}, \quad \dot{p}_i{}^2 = {}^2/_3\,\dot{p}_{mn}\,\dot{p}_{mn}, \quad \sigma_i{}^2 = {}^3/_2\, s_{mn} s_{mn}\,.\end{gathered} \tag{2.2}$$

Here the repetition of indices denotes summation.

We denote the stress state corresponding to a perfectly smooth shell without initial irregularities by σ_i° and call it the basic stress state. Then the stress state of an actual shell in creep will deviate from the basic stress state. Hence the stress components and creep strain rates can be written as follows:

$$s_{mn} = \overset{\circ}{s}_{mn} + \delta s_{mn}, \qquad \dot{p}_{mn} = \dot{p}_{mn}^{\circ} + \delta \dot{p}_{mn}\,. \tag{2.3}$$

We can now rewrite Eqs. (2.1) and (2.2):

$$\begin{gathered}\dot{p}_i^{\circ} + \delta\dot{p}_i = g(\sigma_i^{\circ} + \delta\sigma_i,\ p_i^{\circ} + \delta p_i)(\sigma_i^{\circ} + \delta\sigma_i)\,, \\ \dot{p}_{mn}^{\circ} + \delta\dot{p}_{mn} = {}^3/_2\, g(\sigma_i^{\circ} + \delta\sigma_i,\ p_i^{\circ} + \delta p_i)(\overset{\circ}{s}_{mn} + \delta s_{mn})\,.\end{gathered} \tag{2.4}$$

At the same time, for the basic state Eqs. (2.1) and (2.2) also hold:

$$\dot{p}_i^{\circ} = g(\sigma_i^{\circ},\, p_i^{\circ})\,\sigma_i^{\circ}, \qquad \dot{p}_{mn}^{\circ} = {}^3/_2\, g(\sigma_i^{\circ},\ p_i^{\circ})\,\overset{\circ}{s}_{mn}\,.$$

Now, representing the right-hand sides of Eqs. (2.4) as a series in the neighborhood of the basic state and retaining only the linear terms, we obtain (see [1, 2]

$$\delta\varepsilon^{\cdot}_{mn} - ({}^2/_3 E)^{-1}\delta s^{\cdot}_{mn} = {}^3/_2 g(\sigma_i^\circ, p_i^\circ)\{\delta s_{mn} + \alpha^*_{mn}(Ec\delta p_i + b\delta\sigma_i)\},$$

$$\delta p_i^{\cdot} = g(\sigma_i^\circ, p_i^\circ)\{Ec\delta p_i + (b+1)\delta\sigma_i\}, \qquad (2.5)$$

$$c = \frac{\sigma_i^\circ}{E}\frac{1}{g}\frac{\partial g}{\partial p_i}, \qquad b = \frac{\sigma_i^\circ}{g}\frac{\partial g}{\partial \sigma_i}, \qquad \alpha^*_{mn} = \frac{\overset{\circ}{s}_{mn}}{\sigma_i^\circ}.$$

Introducing the new time variable

$$\tau = E\int_0^t g(\sigma_i^\circ, p_i^\circ)\,dt, \qquad (2.6)$$

we obtain

$$\delta s^{\cdot}_{mn} + \delta s_{mn} = {}^2/_3 E\delta\varepsilon^{\cdot}_{mn} - \alpha^*_{mn}(Ec\delta p_i + b\delta\sigma_i),$$

$$E\delta p_i^{\cdot} = Ec\delta p_i + (b+1)\delta\sigma_i. \qquad (2.7)$$

For the initial moment of time we take Hooke's law

$$\delta\varepsilon_{mn} = {}^3/_2 E^{-1}\delta s_{mn}. \qquad (2.8)$$

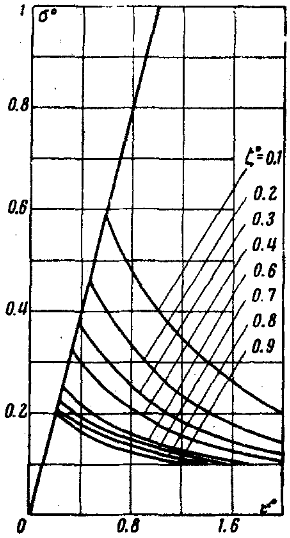

Fig. 1

After integrating Eqs. (2.7) with initial conditions (2.8), we obtain

$$\delta s_{mn} = {}^2/_3 EI\delta\varepsilon_{mn} - \alpha^*_{mn}G\delta\sigma_i. \qquad (2.9)$$

Here

$$I\delta\varepsilon_{mn} = \delta\varepsilon_{mn} - e^{-\tau}\int_0^\tau e^{\tau}\delta\varepsilon_{mn}\,d\tau,$$

$$G\delta\sigma_i = e^{-\tau}\int_0^\tau e^{\tau}\left\{e^{c^*}c\int_0^\tau (b+1)e^{-c^*}\delta\sigma_i\,d\tau + b\delta\sigma_i\right\}d\tau \qquad \left(c^* = \int c\,d\tau\right).$$

For the shell strain increments we use the expressions

$$\delta\varepsilon_{mn} = \delta\varepsilon^c_{mn} + z(\varkappa_{mn} - \varkappa^\circ_{mn}),$$

$$\delta\varepsilon^c_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right) + \frac{1}{2}\frac{\partial w}{\partial x_n}\frac{\partial w}{\partial x_m} - \frac{1}{2}\frac{\partial w^\circ}{\partial x_n}\frac{\partial w^\circ}{\partial x_m} - \frac{w - w^\circ}{R_{mn}}\delta_{mn},$$

$$(x_1 = x, \quad x_2 = y, \quad m = 1, 2, \quad n = 1, 2) \quad u_1 = u, \; u_2 = v.$$

Here $\varkappa_{mn}^\circ$, w° are the initial curvatures and the initial deflection before application of load.

§3. Forces and moments. We now write expressions for the forces and moments referred to the middle surface of the shell

$$M_{11} = \int_{(z)} (2\delta s_{11} + \delta s_{22})\,z\,dz, \ldots,$$

$$T_{11} = \int_{(z)} (2\delta s_{11} + \delta s_{22})\,dz, \ldots. \qquad (3.1)$$

Substituting the stresses (2.9) into Eqs. (3.1) and eliminating the strains, in accordance with (2.10), we obtain (D = Eh$^{3/9}$)

$$M_{11} = DI(\varkappa_{11} + {}^1/_2\varkappa_{22} - \varkappa_{11}^\circ - {}^1/_2\varkappa_{22}^\circ) - \alpha_{11}GM_i,$$

$$M_{22} = DI(\varkappa_{22} + {}^1/_2\varkappa_{11} - \varkappa_{22}^\circ - {}^1/_2\varkappa_{11}^\circ) - \alpha_{22}GM_i, \qquad (3.2)$$

$$M_{12} = {}^1/_2 DI(\varkappa_{12} - \varkappa_{12}^\circ) - \alpha_{12}GM_i,$$

$$M_i = (\alpha_{11} - {}^1/_2\alpha_{22})M_{11} + (\alpha_{22} - {}^1/_2\alpha_{11})M_{22} + 3\alpha_{12}M_{12},$$

$$\alpha_{11} = 2\alpha_{11}^* + \alpha_{22}^*, \qquad \alpha_{22} = 2\alpha_{22} + \alpha_{11}, \qquad \alpha_{12} = \alpha_{12}^*. \qquad (3.3)$$

Similarly, we obtain the specific forces

$$T_{11} = \frac{4Eh}{3}I\left[\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w^\circ}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2 - \right.$$

$$\left. - \frac{1}{4}\left(\frac{\partial w^\circ}{\partial y}\right)^2\right] - \frac{w - w^\circ}{R_{11}} - \frac{w - w^\circ}{2R_{22}}\Big] - \alpha_{11}GT_i, \qquad (3.4)$$

$$T_{22} = \frac{4Eh}{3}I\left[\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w^\circ}{\partial y}\right)^2 - \right.$$

$$\left. - \frac{1}{4}\left(\frac{\partial w^\circ}{\partial x}\right)^2\right] - \frac{w - w^0}{2R_{11}} - \frac{w - w^0}{R_{22}}\Big] - \alpha_{22}GT_i,$$

$$T_{12} = \frac{Eh}{3}I\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w^\circ}{\partial x}\frac{\partial w^\circ}{\partial y}\right] - \alpha_{12}GT_i,$$

$$T_i = (\alpha_{11} - {}^1/_2\alpha_{22})T_{11} - (\alpha_{22} - {}^1/_2\alpha_{11})T_{22} + 3\alpha_{12}T_{12}. \qquad (3.5)$$

For the moment and force increments at loss of stability we have the following relations:

$$M_{11}^* = D(\varkappa_{11}^* + {}^1/_2\varkappa_{22}^*),$$

$$M_{22}^* = D(\varkappa_{22}^* + {}^1/_2\varkappa_{11}^*), \quad M_{12}^* = {}^1/_2 D\varkappa_{12}^*, \qquad (3.6)$$

$$T_{11}^* = \frac{4Eh}{3}\left[\frac{\partial u^*}{\partial x} + \frac{1}{2}\frac{\partial v^*}{\partial y} - \frac{w^*}{2R_{22}} - \frac{w^*}{R_{11}} + 2\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w^*}{\partial y}\right],$$

$$T_{22}^* = \frac{4Eh}{3}\left[\frac{\partial v^*}{\partial y} + \frac{1}{2}\frac{\partial u^*}{\partial x} - \frac{w^*}{R_{22}} - \frac{w^*}{2R_{11}} + 2\frac{\partial w}{\partial y}\frac{\partial w^*}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right],$$

$$T_{12}^* = \frac{Eh}{3}\left[\frac{\partial u^*}{\partial y} + \frac{\partial w^*}{\partial x} + \frac{\partial v^*}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial y}\right].$$

The expressions for T^*_{mn} have been linearized with respect to w^*.

§4. Equations of stability of a flat shell of general form. Introducing the force functions for T_{mn} and T^*_{mn}, we satisfy the first equations of (1.2) and (1.4), while to determine the force functions we set up the strain continuity equations. The remaining equilibrium equations (1.2) and (1.4) are written in terms of displacement, using relations (3.3) and (3.5). After performing all these operations, we finally obtain the following system of equations of the subcritical state:

$$K(w, F) + G[K(w, F) + T(w)] = 0,$$

$$\nabla^2\nabla^2 F + \Lambda_1^2 GF = -EhI\left[\frac{1}{R_{11}}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w^\circ}{\partial y^2}\right) + \frac{1}{R_{22}}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w^\circ}{\partial x^2}\right) + \right.$$

$$\left. + \frac{L(w, w)}{2} - \frac{L(w^\circ, w^\circ)}{2}\right], \qquad (4.1)$$

$$K(w, F) = D\nabla^2\nabla^2 I(w - w^\circ) -$$
$$- h\sigma_i^\circ \Lambda w - L(w, F) - \frac{1}{R_{22}}\frac{\partial^2 F}{\partial x^2} - \frac{1}{R_{11}}\frac{\partial^2 F}{\partial y^2},$$
$$L(w, F) = + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y},$$
$$T(w) = -{}^3/_4 D\Lambda^2 I(w - w^\circ)$$

and the system of stability equations

$$\nabla^2\nabla^2 F^* = -Eh\left(\frac{1}{R_{11}}\frac{\partial^2 w^*}{\partial y^2} + \frac{1}{R_{22}}\frac{\partial^2 w}{\partial x^2} + L(w^*, w)\right),$$
$$D\nabla^2\nabla^2 w^* - h\sigma_i^\circ \Lambda w^* - \frac{1}{R_{22}}\frac{\partial^2 F^*}{\partial x^2} -$$
$$- \frac{1}{R_{11}}\frac{\partial_2 F}{\partial y^2} - L(w, F^*) - L(w^*, F) = 0, \tag{4.2}$$
$$\Lambda = \alpha_{11}\frac{\partial^2}{\partial x^2} + 2\alpha_{12}\frac{\partial^2}{\partial x\,\partial y} + \alpha_{22}\frac{\partial^2}{\partial y^2},$$
$$\Lambda_1 = \alpha_{11}\left(\frac{\partial^2}{\partial y^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) + \alpha_{22}\left(\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial y^2}\right) + 3\alpha_{12}\frac{\partial}{\partial x\,\partial y}.$$

§5. Closed cylindrical shell in axial compression. Let us consider the stability of a closed cylindrical shell subjected to uniform axial compression ($R_{11} = \infty$, $R_{22} = R$, $\alpha = -1$, $\alpha_{12} = \alpha_{22} = 0$) using for this purpose the simple creep law

$$p_i^\cdot = B\sigma_i. \tag{5.1}$$

Then the system of equations (4.1) becomes

$$\nabla^2\nabla^2 F = -\frac{Eh}{R} I\left[\frac{\partial^2(w - w^\circ)}{\partial x^2} + \frac{R}{2}L(w, w) - \frac{R}{2}L(w^\circ, w^\circ)\right],$$
$$D\nabla^2\nabla^2 I(w - w^\circ) + T^\circ\frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial^2 F}{\partial x^2} - L(w, F) = 0.$$

To solve the problem of selecting the mode of deflection before and after loss of stability and also the initial mode of deflection, we turn to the nonlinear problem of the postbuckling behavior of a cylindrical shell, which has been studied by a number of workers [3–8]. These investigations employ various representations of the postbuckling mode of deflection, but they all contain a common element which can be written as follows:

$$w_1 = f_1 \sin\frac{m\pi x}{L}\sin\frac{ny}{R} + f_2\sin^2\frac{m\pi x}{L} + f_0.$$

Using this expression for the postbuckling deflection as a basis, we may write

$$w^\circ = h\zeta^\circ\sin^2\frac{m\pi x}{L}, \qquad w = h\zeta(\tau)\sin^2\frac{m\pi x}{L} + h\zeta_1(\tau), \tag{5.3}$$
$$w^* = h\zeta^*\sin\frac{m\pi x}{L}\sin\frac{ny}{R}.$$

As a result, after loss of stability we have

$$w_1 = h\zeta^*\sin\frac{m\pi x}{L}\sin\frac{ny}{R} + h\zeta\sin^2\frac{m\pi x}{L} + h\zeta_1.$$

Solving the problem in its present form does not yield 5*. Integrating the system of equations (5.2), we obtain

$$\zeta(\tau) = (k + 1)\zeta^\circ e^{k\tau}, \qquad T_{22} = \frac{Eh^2}{R}k\zeta^\circ e^{k\tau}\cos\frac{2m\pi x}{L}, \tag{5.4}$$
$$k = \frac{\sigma}{l_1 - \sigma}, \qquad \sigma = \frac{T^\circ R}{Eh^2}, \qquad l_1 = \frac{4\eta\vartheta^2}{9} + \frac{1}{4\eta\vartheta^2}, \qquad \vartheta = \frac{m\pi R}{Ln},$$
$$\eta = \frac{h}{R}n^2.$$

Eliminating w, F, and F* from the second equation of the system (4.2) and integrating it by the Bubnov-Galerkin method, we get the equation for the critical time parameter $\tau = \tau_*$

$$4l_3(k + 1)^2\zeta^{\circ 2}e^{2k\tau} - (1 + k)l_4\zeta^\circ e^{k\tau} + l_2 - \sigma = 0,$$
$$l_2 = \frac{(1 + \vartheta^2)^2}{9\vartheta^2}\eta + \frac{\vartheta^\circ}{\eta(1 + \vartheta^2)^2},$$
$$l_3 = \eta\vartheta^2\left[\frac{1}{(1 + \vartheta^2)^2} + \frac{1}{(1 + 9\vartheta^2)^2}\right], \qquad l_4 = \frac{4\vartheta^2}{(1 + \vartheta^2)^2} + \frac{k}{(k + 1)^2\vartheta^2}.$$

Figure 1 gives the dependence of the critical axial strain of the basic state on the initial stresses and the amplitude of the initial irregularities. The following notation has been used:

$$\sigma^\circ = \sigma/\sigma^+, \qquad \varepsilon^\circ = \sigma^\circ(1 + \tau_*)$$

where σ^+ is the upper critical stress for a perfectly smooth elastic shell ($\sigma^+ = 0.605$). The wave parameters ϑ and η were selected from the condition of minimum τ_*. Values of the critical strain parameter ε° are plotted along the abcissa axis.

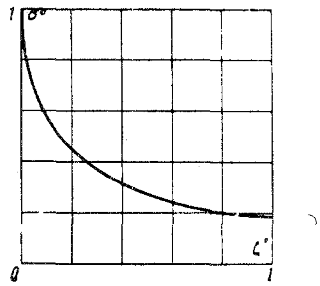

Fig. 2

At $\tau = 0$ Eq. (5.5) leads to the solution of the problem of stability of an elastic shell with account for initial irregularities. Figure 2 gives the dependence of the critical stress for an elastic shell on the amplitude of the initial deflection ζ°. The graph shows that the magnitude of the critical stress changes sharply at very small initial deflections. This suggests that in tests it is very difficult to obtain $\sigma^+ = 0.605$ or, what is the same thing, $\sigma^\circ = 1$.

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